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VANISHING THEOREMS FOR COHOMOLOGIES OF AUTOMORPHIC VECTOR BUNDLES

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1. Introduction

Let M be a compact, irreducible, locally hermitian symmetric space of noncompact type. If M is not a Riemann surface, then Calabi and Vesentini have shown in ([4]), that the complex structure on M is infinitesimally rigid, i.e., they show the vanishing of $H^1(M, \Theta_M)$, where Θ_M is the tangent sheaf of germs of holomorphic vector fields on M. Their method involves the construction of an 'auxiliary expression', which is simplified in two different ways, to obtain a quadratic form involving curvature terms. The desired vanishing is reduced then to proving that the quadratic form is positive definite. One obtains criteria for the vanishing of the cohomology groups $H^*(M, \Theta_M)$, which depend on the curvature properties of M and not on the lattice defining M.

Based on their method, Weil showed (see [17]) that an irreducible, cocompact lattice Γ in a real semisimple Lie group G without compact or three dimensional factors is rigid, i.e., any 'nearby' deformations of Γ inside G are equivalent. This amounts to showing the vanishing of $H^1(\Gamma, Ad)$, where Ad is the adjoint representation of G on its Lie algebra. Matsushima refined this method to show vanishing of Betti numbers of M below some degree ([10]).

Our main result is to give a criterion for the vanishing of cohomologies of "automorphic" vector bundles, generalising the results of Calabi-Vesentini and Matsushima. More generally we give a criterion for the vanishing of $\bar{\partial}$-cohomology (or $(\underline{q}, K^{\mathbf{C}})-$cohomology) of unitary $\underline{g}$- modules with coeffecients in a K- module. The terms involved can be calculated explicitly in terms of the dominant weight of the reprsentation defining the automorphic vector bundle and the curva-

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ture constants of M.

Apart from Calabi-Vesentini and Matsushima, vanishing theorems for cohomologies of automorphic vector bundles have been proved by many authors, among them Narasimhan-Okamoto, Griffiths, Schmid, Hotta and Parthasarthy. See for instance ([15]). However they all involve some regularity assumptions on the dominant weight of the reprsentation defining the automorphic vector bundle and are not applicable in general.

Our main application of and motivation for the vanishing theorem on cohomology, is to generalize the rigidity theorems of Calabi-Vesentini. We show for a large class of compact, locally homogeneous Kahlerian spaces, that the complex structure on them is infinitesimally rigid. These are precisely the spaces which are fibered over a locally hermitian symmetric domain. To show the infinitesimal rigidity of these spaces, we use the Leray spectral sequence for the fibering, to reduce the question to one concerning the vanishing of the first cohomolgy of certain automorphic vector bundles on the associated locally hermitian symmetric domain.

Let $\tilde{M}$ be the universal cover of M. Let G be the group of isometries of the symmetric space $\tilde{M}$. Let $\underline{g}$ be the Lie algebra of G, $U(\underline{g}^{\mathbf{C}})$ be the universal enveloping algebra of the complexification of $\underline{g}$ and $\underline{Z}$ be the center of the $U(\underline{g}^{\mathbf{C}})$. If E is an automorphic vector bundle on M, it is observed in ([5]), that there is a natural action of $\underline{Z}$ on the Dolbeault complex $\mathcal{D}$ of $(0,p)$-forms $(0 \leq p \leq \dim M)$, and hence on the cohomology $H^*(M,E)$. Faltings in fact shows that each $Z \in \underline{Z}$ acts as a scalar on $H^p(M,E)$, and morever determines the corresponding homomorphism of $\underline{Z}$ into $\mathbf{C}$. In view of the homogeneous nature of the $\bar{\partial}$-Laplacian (with respect to an appropriate metric on E), the space of harmonic $(0,p)$- forms is $\underline{Z}$-stable and hence harmonic p-forms are eigenforms for the action of $\underline{Z}$ on $\mathcal{D}$.

Thus we can view harmonic forms as eigenforms of the Casimir of $\underline{g}^{\mathbf{C}}$. This allows us to bypass the complicated calculations of Laplacians on bundles (see for instance [6]) and instead work in a group theoretic framework. We believe that the expressions are more amenable to calculations in a group theoretic framework. The vanishing result we obtain in fact asserts that under suitable conditions on the bundle E, nonzero eigenforms for $\underline{Z}$ corresponding to these eigenvalues do not exist.

Briefly our proof is as follows: Following the lines of Matsushima's proof as adapted by Borel-Wallach, we obtain a quadratic form involving norm and the derivatives of the eigen-form. We then use the result of Faltings, to express the norm of the form in terms of the norms of the derivatives of the form. We obtain then a quadratic form on $\underline{p} \otimes \underline{p}$, whose positivity ensures vanishing results on cohomology.

Remark. The method of Calabi-Vesentini has been further exploited by Corlette, Siu, Jost, Yau, Mok and Yeung to obtain archimedean superrigidity of lattices in semisimple Lie groups except those in $SO(n,1)$ and $SU(n,1)$. See ([9], [14]). However apart from Calabi-Vesentini and Matsushima this method does not seem to have been used to compute the cohomologies of automorphic vector bundles. For the relationship of these cohomology groups with arithmetic, we refer to the article of M. Harris ([7]).

2. Preliminaries

2.1. Let G be a real, semisimple, connected linear Lie group without compact factors. Let K be a maximal compact subgroup of G. We assume that $\tilde{M} = G/K$ is a bounded hermitian symmetric domain. Let $T \subseteq K$ be a compact Cartan subgroup of G. Denote the Lie algebras of left invariant vector fields on G, K, T by $\underline{g}$, $\underline{k}$, $\underline{t}$ respectively. Let $\underline{g}^{\mathbf{C}}$, $\underline{k}^{\mathbf{C}}$, $\underline{t}^{C}$ denote respectively their complexifications. Let $G^{\mathbf{C}}$ be the complexification of G with Lie algebra $\underline{g}^{\mathbf{C}}$. Let $K^{\mathbf{C}}$ denote the complex subgroup of $G^{\mathbf{C}}$ corresponding to the Lie subalgebra $\underline{k}^{\mathbf{C}}$ of $\underline{g}^{\mathbf{C}}$. Let B denote the Killing form on $\underline{g}^{\mathbf{C}}$. Let $U(\underline{g}^{\mathbf{C}})$ denote the universal enveloping algebra of $\underline{g}^{\mathbf{C}}$. Let $\underline{Z}$ denote the centre of the universal enveloping algebra $U(\underline{g}^{\mathbf{C}})$.

Note. For all group theoretical facts we refer to ([8]).

One has the Cartan decomposition $\underline{g} = \underline{k} \oplus \underline{p}$, where $\underline{p}$ is the orthogonal complement of $\underline{k}$ inside $\underline{g}$. There is a natural identification of $\underline{p}$, with the tangent space at the identity coset of G/K. The complexification $\underline{p}^{\mathbf{C}}$ of $\underline{p}$ splits canonically into two $\underline{k}$ invariant subspaces $\underline{p}^{\mathbf{C}} = \underline{p}^{+} \oplus \underline{p}^{-}$, such that $\underline{p}^{+}$ and $\underline{p}^{-}$ are abelian subalgebras of $\underline{g}^{\mathbf{C}}$. Let P^{+} and P^{-} denote the subgroups of $G^{\mathbf{C}}$ corresponding respectively to the subalgebras $\underline{p}^{+}$ and $\underline{p}^{-}$ of $\underline{g}^{\mathbf{C}}$. $Q = P^{-}K^{\mathbf{C}}$ is then a (complex) parabolic subgroup of $G^{\mathbf{C}}$ with Lie algebra $\underline{q} = \underline{p}^{-} \oplus \underline{k}^{\mathbf{C}}$.

$P^+K^{\mathbf{C}}P^-$ is an open subset of $G^{\mathbf{C}}$ containing G and $G \cap K^{\mathbf{C}}P^- = K$. Thus $\tilde{M} = G/K$ can be embedded as a open subset in the compact complex manifold $G^{\mathbf{C}}/Q$ and let the complex structure on G/K be the one induced by this open immersion. $\underline{p}^+$ (resp. $\underline{p}^-$) then corresponds to the space of holomorphic (resp. antiholomorphic) tangent vectors at the identity coset of G/K.

Let Δ denote the collection of roots of the pair $(\underline{g}^{\mathbf{C}},\ \underline{t}^{\mathbf{C}})$. Let Δ_c (resp. Δ_n) denote the set of compact (resp. noncompact) roots i.e. those $\alpha \in \Phi$ for which $\underline{g}^\alpha \subset \underline{k}^{\mathbf{C}}$ (resp. $\underline{g}^\alpha \subset \underline{p}^{\mathbf{C}}$). We choose an ordering of the roots, such that $\underline{p}^+$ is the span of the root spaces corresponding to the noncompact positive roots. Let ρ denote half the sum of positive roots of $\underline{g}^{\mathbf{C}}$.

The smooth tangent bundle to G/K is a G-equivariant bundle and there is a G-equivariant isomorphism $T(G/K) \simeq G \times_K \underline{p}$. The Killing form B on $\underline{g}$, restricts to a positive definite form on $\underline{p}$ (and is negative definite on $\underline{k}$). This gives rise to a G-invariant metric on G/K. Let R denote the curvature tensor on G/K. For $X, Y \in T_{eK}(G/K)$, $R(X,Y) \in \mathrm{End}(T_{eK}(G/K))$. It is well known that under the natural identification $T_{eK}(G/K) \simeq \underline{p}$, R is given by

$$R(X,Y)Z = [[Y,X],Z] \qquad (X,Y,Z \in \underline{p}) \tag{1}$$

Let $\{X_i\}_{1\leq i \leq d}$ form an orthonormal basis of $\underline{p}$. We choose an orthogonal basis $\{X_a\}_{d<a\leq n}$ of $\underline{k}$, with $B(X_a, X_a) = -1$ and such that each X_a is either in the center of $\underline{k}$ or is in one of the simple ideals of $\underline{k}$. Define the structure constants, c_{ij}^a, c_{aj}^i $1 \leq i,\ j \leq d,\ d \leq a \leq n$ by $[X_i, X_j] = \sum_a c_{ij}^a X_a$ and $[X_a, X_i] = \sum_j c_{ai}^j X_j$. From the invariance of the Killing form it follows that

$$c_{ij}^a = -c_{ji}^a = c_{aj}^i \tag{2}$$

Define $R_{ijkl} = B([X_i,X_j],[X_k,X_l])$. In terms of the structure constants,

$$R_{ijkl} = -\sum_a c_{ij}^a c_{kl}^a \tag{3}$$

G acts by left and right multiplication on itself. Denote the corresponding action on the functions on G by L and R i.e. for $g, g' \in$

G, $L(g)(f)(g') = f(g^{-1}g')$, $R(g)(f)(g') = f(g'g)$. We continue to denote by L and R, the induced actions of $U(\underline{g}^{\mathbf{C}})$ on smooth functions on G as well. The left and right actions are compatible in the following sense: for $X \in \underline{g}^{\mathbf{C}}$, let $X^t = -X$. This extends to an antiautomorphism $X \mapsto X^t$ of $U(\underline{g}^{\mathbf{C}})$. Then any $Z \in \underline{Z}$ satisfies, $L(Z) = R(Z^t)$.

Let Γ be an irreducible, torsion-free, cocompact lattice in G. Let $M = \Gamma\backslash G/K$. We will assume that Γ is torsion free in order to ensure that M be smooth. We want to study the cohomology of homogeneous vector bundles on M.

2.2. We now describe a class of holomorphic vector bundles on M. Let σ be a representation of K on a vector space V. We continue to denote by σ the corresponding holomorphic representation of Q on V, which is trivial on P^+. One can form the holomorphic $G^{\mathbf{C}}$–equivariant vector bundle on G^C/Q, associated to the holomorphic principal Q–fibration $G^{\mathbf{C}} \to G^{\mathbf{C}}/Q$. Restrict this bundle to the open set $\tilde{M}$ and take the quotient by the action of Γ to get a holomorphic vector bundle $E(\sigma)$ on M. We call such bundles as 'automorphic vector bundles' in the sequel.

We note that as C^∞-vector bundles, one has a natural isomorphism of $E(\sigma)$ with the bundle associated to the principal K-bundle $\Gamma\backslash G \to \Gamma\backslash G/K$ and the reprsentation σ (restricted to K) of K. See ([11]). We can then define a G invariant metric on the bundle $E(\sigma)$ by taking any metric on V on which K acts as isometries.

Example. The holomorphic tangent sheaf Θ of M is isomorphic to the automorphic vector bundle associated to the representation of $K^{\mathbf{C}}$ on $\underline{p}^+$. (We will identify a vector bundle with its associated locally free sheaf of germs of sections). Since one can identify $(\underline{p}^+)^*$ and $\underline{p}^-$ by the Killing form, the cotangent bundle Ω^1 of M is the automorphic vector bundle associated to the representation of $K^{\mathbf{C}}$ on $\underline{p}^-$.

2.3. Let $A^{p,q}(M,\ E_\sigma)$ denote the space of smooth forms of type (p,q) on M with values in the bundle E_σ. $A^{p,q}(M, E_\sigma)$ is isomorphic to the subspace of $C^\infty(\Gamma\backslash G) \otimes V \otimes \bigwedge^p \underline{p}^- \otimes \bigwedge^q \underline{p}^+$ defined by the following condition: (see [12, page 9]) for $X \in \underline{k}^{\mathbf{C}}$ and $\eta \in C^\infty(\Gamma\backslash G) \otimes V \otimes \bigwedge^p \underline{p}^- \otimes \bigwedge^q \underline{p}^+$,

$$(R \otimes \sigma \otimes ad_-^p \otimes ad_+^q)(X)\eta = 0 \tag{4}$$

where ad_-^p (resp. ad_+^q) denotes the action of $\underline{k}^{\mathbf{C}}$ on $\bigwedge^p \underline{p}^-$ (resp. $\bigwedge^q \underline{p}^+$).

The cohomology groups admit an interpretation in terms of relative Lie algebra cohomology. Let $\{X_\alpha\}_{1\leq\alpha\leq d}$ be a basis of $\underline{p}^+$, with $\{X_{\overline{\alpha}}\}_{1\leq\alpha\leq d}$ the corresponding dual basis of $\underline{p}^-$.

We have the differential operator $\overline{\partial} : A^{0,q}(M, E_\sigma) \to A^{0,q+1}(M, E_\sigma)$ (see [12, page 16]) :

$$\overline{\partial} = \sum_{\alpha=1}^{d} R(X_{\overline{\alpha}}) \otimes 1 \otimes \epsilon(X_\alpha)$$

Since $\sigma(X_{\overline{\alpha}}) = 0$, $\overline{\partial}$ is just the differential in the complex computing the relative $(\underline{q}, K^{\mathbf{C}})$ cohomology of the $(\underline{q}, K^{\mathbf{C}})$ -module $C^\infty(\Gamma\backslash G) \otimes V$. Hence

$$\begin{aligned} H^q(M, E_\sigma) &\simeq H^{0q}(M, E_\sigma, \overline{\partial}) \simeq H^q(A^{0*}(M, E_\sigma), \overline{\partial}) \\ &\simeq H^q(\underline{q}, K^{\mathbf{C}}, C^\infty(\Gamma\backslash G) \otimes V). \end{aligned}$$

From (4), we see that the center $\underline{Z}$ of the universal enveloping algebra of $U(\underline{g}^{\mathbf{C}})$ acts on $A^{p,q}(M, E_\sigma)$ (via R or L). Morever from the formulas for d and $\bar{\partial}$, we see that the $\underline{Z}$ action commutes with $\bar{\partial}$. Hence there is an action of $\underline{Z}$ on the cohomology groups of the vector bundles considered above. This action will be basic to our study of the cohomology of automorphic vector bundles. In fact our vanishing theorems state that under suitable conditions on the highest weight of the $\underline{k}^{\mathbf{C}}$ module, certain eigenforms of $\underline{Z}$ vanish.

We note that the left (right) translations of G, L (resp. R), acting only on the first factor G of $G \times V$, act unitarily on the space of sections of $E(\sigma)$. Thus for any $X \in \underline{g}^{\mathbf{C}}$, we have $L(X)^* = -L(X)$ (similarly for $R(X)$), where $*$ denotes the adjoint with respect to the metrics considered above. Hence for any $X \in U(\underline{g}^{\mathbf{C}})$, $L(X)^* = L(X^t)$, where $X \mapsto X^t$ is the involution on $U(\underline{g}^{\mathbf{C}})$ considered above.

With respect to the metric defined above one can define the adjoint of $\bar{\partial}^*$ of $\bar{\partial}$ and the Laplacian $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. As usual one can do a harmonic theory and represnt the cohomology by harmonic forms.

Proposition 1. *The left (or right) action L of $\underline{Z}$ described above leaves stable the space of harmonic forms.*

Proof. Let $Z \in \underline{Z}$. We have seen above that $Z \circ \bar{\partial} = \bar{\partial} \circ Z$ and

$L(Z)^* = L(Z^t)$. Let η, ω be smooth forms with values in $E(\sigma)$. Then

$$\begin{aligned} < L(Z)\bar{\partial}^*\eta, \omega > &= < \bar{\partial}^*\eta, L(Z^t)\omega > = < \eta, \bar{\partial} L(Z^t)\omega > \\ &= < \eta, L(Z^t)\bar{\partial}\omega > = < L(Z)\eta, \bar{\partial}\omega > \\ &= < \bar{\partial}^* L(Z)\eta, \omega > \end{aligned}$$

Hence $L(Z)\bar{\partial}^* = \bar{\partial}^* L(Z)$ and so $L(Z) \circ \Delta_{\bar{\partial}} = \Delta_{\bar{\partial}} \circ L(Z)$. Consequently the space of harmonic forms is $\underline{Z}$ stable, the $\underline{Z}$ action being the same as in the cohomology.

2.4. We now recall a theorem of Faltings describing the action of the centre of the universal enveloping algebra on the cohomology groups $H^*(M, E(\sigma))$.

Let W be the Weyl group of $(\underline{g}^{\mathbf{C}}, \underline{t}^{\mathbf{C}})$. By a theorem of Harishchandra ([16, chapter 3]), one knows that $\underline{Z}$ is isomorphic to the $W-$ invariants of the symmetric algebra $S(\underline{t}^{\mathbf{C}})$ of $\underline{t}^{\mathbf{C}}$. Let $\theta : \underline{Z} \to S(\underline{t}^{\mathbf{C}})$ be the Harishchandra isomorphism. The symmetric algebra on $\underline{t}^{\mathbf{C}}$ can be identified with the space of polynomial functions on $(\underline{t}^{\mathbf{C}})^*$. Given a complex linear form λ on $\underline{t}^C$, let e_λ be the evaluation map at λ. For any complex linear form λ on $\underline{t}^C$, define a character $\chi_\lambda : \underline{Z} \to \mathbf{C}$, by $\chi_\lambda = e_{\lambda+\rho} \circ \theta$. χ_λ describes the action of $\underline{Z}$ on the Verma module with highest weight λ. $\chi_\lambda = \chi_\mu$ iff $\lambda + \rho$ and $\mu + \rho$ are conjugate under the Weyl group W.

Let μ be the highest weight of the $K^{\mathbf{C}}$ - representation σ. μ is a linear form on $\underline{t}^{\mathbf{C}}$. The theorem of Faltings says the following: ([5, page 77])

Theorem 2 (Faltings). *$\underline{Z}$ acts on the cohomology groups $H^*(M, E_\sigma)$ and hence also on the space of harmonic forms on M with values in E_σ, via the left action L, by the infinitesimal character χ_μ.*

For the sake of completeness we outline a proof of the theorem. Let W be a unitary $(\underline{g}, K)$-module, possessing an infinitesimal character χ_W. The cohomology group $H^q(\underline{q}, K^{\mathbf{C}}, W \otimes V)$, can be thought of as Ext groups in the category of $(\underline{q}, K^{\mathbf{C}})$-modules. We have

$$H^q(\underline{q}, K^{\mathbf{C}}, W \otimes V) = \operatorname{Ext}^q_{(\underline{q}, K^{\mathbf{C}})}(V^*, W)$$

By general homological algebra, it is enough to know the Z-action when $q = 0$. Inducing to $(\underline{g}, K^{\mathbf{C}})$, we have the isomorphism

$$\operatorname{Hom}_{(\underline{q}, K^{\mathbf{C}})}(V^*, W) = \operatorname{Hom}_{(\underline{g}, K^{\mathbf{C}})}(U(\underline{g}^{\mathbf{C}}) \otimes_{\underline{U}(q)} V^*, W)$$

By Poincare-Birkhoff-Witt, it is easy to see that the cohomology groups vanish unless $\chi_W = \chi_\mu$ (note that $\underline{q} = \underline{k}^{\mathbf{C}} + \underline{p}^-$) and that Z acts by χ_μ. This finishes the proof of the theorem.

2.5. We now make a few remarks about Casimir operators. Let $\underline{g}'$ be a complex reductive Lie algebra. Let B' be a nondegenerate invariant form on $\underline{g}'$. Let $(Y_i)_{1\leq i\leq n}$ and $(Y_i')_{1\leq i\leq n}$ be bases of $\underline{g}'$ dual with respect to B' i.e. $B'(Y_i, Y_j') = \delta_{ij}$. Define the Casimir element $C_{B'} = \sum_{i=1}^n Y_i Y_i'$ in $U(\underline{g})$, with respect to the invariant form B'. Then $C_{B'}$ is a central element of $U(\underline{g})$ and is independent of the choice of the pair of bases. Let $\underline{t}'$ be a Cartan subalgebra of $\underline{g}'$ and fix an order on the set of roots $(\underline{g}', t')$. Let $2\rho'$ be the sum of the positive roots. Let λ' be a linear form on $\underline{t}'$ and let $M_{\lambda'}$ be the corresponding Verma module. By choosing the bases to consist of root vectors and by analysing the action on the highest weight vector of $M_{\lambda'}$, we see easily that

$$\chi_{\lambda'}(C_{B'}) = B'(\lambda', \lambda') + 2B'(\lambda', \rho') \tag{5}$$

where we continue to denote by B' the restriction of B' to Hom $(t', \mathbf{C})$. Since any highest weight module $V_{\lambda'}$ with highest weight λ' is a quotient of $M_{\lambda'}, C_{B'}$ continues to act on V_λ, by $\chi_{\lambda'(C_{B'})}.Id$.

Remark. Let C_G denote the Casimir of $\underline{g}^{\mathbf{C}}$, with respect to the Killing form B of $\underline{g}^{\mathbf{C}}$. $C_G \in \underline{Z}$ and $C_G^t = C_G$. Thus the Casimir C_G of $\underline{g}^{\mathbf{C}}$ acts (left or right) via $\chi_\mu(C_G)$ on the cohomology groups $H^*(M, E(\sigma))$.

2.6. Remark. When the representation σ of Q, is the restriction of a representation of $G^{\mathbf{C}}$, then a vector bundle can be constructed on M as in the case of automorphic bundles. These bundles are flat and the computation of cohomology of the corresponding local system can be reduced to the automorphic case. Let V be an irreducible $\underline{g}$-module and let v be a highest weight vector in V. Let S be the $\underline{k}^{\mathbf{C}}$-span of v. Let η be a $(0,q)$-form on M with values in V, harmonic with respect to the Laplacian of d, constructed with respect to the natural metric. Then it is easy to see from the formulas given in ([11] (Lemma 5.1, page 409)), that η takes values in the subspace S. By appealing then to the theorems of Kuga and Faltings we see that in order to prove vanishing theorems for the cohomology groups in the flat case, we are reduced to proving the vanishing theorems for forms with values in S, which have the right eigenvalue with respect to the Casimir. This allows us to recover the theorem of Weil from the theorem of Calabi and Vesentini. This is just an illustration of Eichler-Shimura theory. See ([12, Theorem 7.1]).

3. A vanishing theorem

We prove our main result (a vanishing theorem for cohomology of automorphic vector bundles) in this section. As has already been remarked, the idea of the proof goes back to Calabi and was refined by Weil and Matsushima. We follow the infinitesimal approach given by Borel and Wallach. ([2] (Chapter II, Section 8))

3.1. Let W be a unitary $\underline{g}$-module. We denote by R the action of $\underline{g}$ on W. For example R can be the right regular representation of G on the space $C^\infty(\Gamma\backslash G)$ of smooth functions on $\Gamma\backslash G$. Let σ be a representation of K on a vector space V, with highest weight μ, with respect to the order on $i\underline{t}$ introduced earlier.

Notation: On the spaces V, and $\underline{p}^+$ we give the metrics given in ( ). On the spaces naturally associated to these spaces, we equip them with the natural metrics constucted out of the metrics given above. In all cases we will continue to denote the metric by $<.,.>$. We will always assume that when we are summing over the variables $i,\ j,\ k,\ l$ the sum will range from 1 to d , and when we are summing over indices $a,\ b$, the indices will range over $d+1$ to n.

Denote by $m = R\otimes\sigma$ the action of $\underline{k}$ on $W\otimes V$. Let η be an element of $W\otimes V\otimes\bigwedge^q\underline{p}^+$, satisfying the equivariance condition (4)

$$(m\otimes ad_+^q)(X)\eta = 0 \ \text{ for } X\in \underline{k}^C \tag{6}$$

If W is $C^\infty(\Gamma\backslash G)$, η can be thought of as a smooth $(0,q)$ form on $\Gamma\backslash G/K$ with values in the bundle E_σ.

Consider the auxiliary expression

$$\Phi(\eta) = \frac{(q-1)!}{2}\sum \parallel m([X_i,X_j])\eta \parallel^2$$

We will simplify $\Phi(\eta)$ in two different ways, one thinking of $W\otimes V\otimes \bigwedge^q p^+$ as a $\underline{k}^{\mathbf{C}}-$ module, another in terms of the curvature components of G/K. For harmonic η, we then construct a quadratic form in the derivatives of η.

3.2. Denote by L, the invariant bilinear form on $\underline{k}^{\mathbf{C}}$ defined by

$$L(X,Y) = tr(ad_{\underline{p}}X ad_{\underline{p}}Y) \quad (X, Y \in \underline{k}^{\mathbf{C}}).$$

By our choice of bases, the $\{X_a\}'s$ form an orthogonal basis with respect to L. Since $\underline{g}$ has no compact factors, $\underline{k}$ acts faithfully on $\underline{p}$. Moreover the eigenvalues of $adX(X \in \underline{k})$ are purely imaginary. Hence L is negative definite. For X_a in the centre of $\underline{k}^{\mathbf{C}}$, we have

$$L(X_a, X_a) = B(X_a, X_a) = -1 \tag{7}$$

Since L is an invariant form on $\underline{k}$, on each of the simple ideals $\underline{k}_i$ of $\underline{k}$, L is a (positive) multiple A_i of the Killing form B restricted to k_i. If $\underline{g}$ is a direct sum of ideals $\underline{g}_r$, then the forms B and L of $\underline{g}$ restrict to the corresponding forms B and L of $\underline{g}_r$. Thus the constants A_i depend only on the simple algebra $\underline{g}_r$. The constants A_i can be explicitly calculated and we have $0 < A_i < 1$. Let $A = \min_i A_i$. For the values of A_i, refer to Table 1.

In terms of the structure constants,

$$L(X_a, X_b) = -\sum_{i,j} c_{ij}^a c_{ij}^b \tag{8}$$

Define a Casimir operator C_L with respect to the invariant form L by

$$C_L = \sum_a L(X_a, X_a) X_a^2 \tag{9}$$

We will now simplify $\Phi(\eta)$ using $\underline{k}^C$-equivariance. Using (6), we obtain

$$\begin{aligned}
\Phi(\eta) &= \frac{(q-1)!}{2} \sum_{i,j,a} (c_{ij}^a)^2 \parallel m(X_a)\eta \parallel^2 \\
&= -\frac{(q-1)!}{2} \sum_a L(X_a, X_a) \parallel m(X_a)\eta \parallel^2 \\
&= -\frac{(q-1)!}{2} \sum_a L(X_a, X_a) < ad_+^q(X_a)\eta,\ ad_+^q(X_a)\eta > \\
&= \frac{(q-1)!}{2} < ad_+^q(C_L)\eta,\ \eta > \qquad (10)
\end{aligned}$$

3.3. Now we transform $\Phi(\eta)$ using the formula $[X_i, X_j] = \sum_a c_{ij}^a X_a$ on only one term of the scalar product.

$$\Phi(\eta) = \frac{(q-1)!}{2} \sum_{i,j,a} c_{ij}^a < m(X_a)\eta,\ m[X_i, X_j]\eta >$$

Using the equivariance of η under $\underline{k}^C$ given by (6), we obtain

$$\Phi(\eta) = -\frac{(q-1)!}{2} \sum_{i,j,a} c_{ij}^a < ad_+^q(X_a)\eta,\ m[X_i, X_j]\eta >$$

(11) Set $\Phi_1(\eta) = -\frac{(q-1)!}{2} \sum_{i,j,a} c_{ij}^a < ad_+^q(X_a)\eta,\ R[X_i, X_j]\eta >$

(12) and $\Phi_2(\eta) = -\frac{(q-1)!}{2} \sum_{i,j,a} c_{ij}^a < ad_+^q(X_a)\eta,\ \sigma[X_i, X_j]\eta >$

(13) Then $\Phi(\eta) = \Phi_1(\eta) + \Phi_2(\eta)$

$\Phi_1(\eta)$ contains the differentiation term only and behaves like the expression for the trivial vector bundle, i.e. the case of constant coefficients. $\Phi_1(\eta)$ will be simplified in terms of the curvature coefficients R_{ijkl}.

$\Phi_2(\eta)$ involves only linear terms and can be written in terms of C_L.

3.4. We now simplify $\Phi_1(\eta)$. We remark that we have identified $\underline{p}^{\mathbf{C}}$ and $(\underline{p}^{\mathbf{C}})^*$ by means of the Killing form restricted to $\underline{p}$. $(\underline{p}^{\mathbf{C}})^*$ is contragredient to the representation of $\underline{k}^{\mathbf{C}}$ on $\underline{p}^{\mathbf{C}}$.

Let $J = \{j_1, \cdots j_q\}$ be a subset of cardinality q of $\{1, \cdots d\}$, and let X_J denote the corresponding element in $\bigwedge^q \underline{p}$ (upto a sign). If η, ω are q-forms on M with values in $E(\sigma)$ then the inner product can be written as

$$< \eta,\ \omega >= \sum_J < \eta(X_J),\ \omega(X_J) >$$

where J runs through subsets of cardinality q of $\{1, ..., d\}$. (Note that this is well defined, since the arbitrariness regarding sign disappears on taking the product). If η, ω are thought of as elements in $W \otimes V \otimes \bigwedge^q \underline{p}$, then $\eta(X_J)$, $\omega(X_J)$ will stand for the coeffecient (with values in $W \otimes V$)

with respect to the basis X_J of $\bigwedge^q \underline{p}$. Thus,

$$\Phi_1(\eta) = -\frac{(q-1)!}{2} \sum_{i,j,q,J} c_{ij}^a < ((ad_+^q(X_a)\eta)(X_J),\ (R[X_i, X,]\eta)(X_J) >$$
$$= -(2q)^{-1} \sum_{\substack{i,j,a \\ j_1,\cdots j_q}} c_{ij}^a < (ad_+^q(X_a)\eta)_{j_1,\cdots j_q},\ (R[X_i, X_j]\eta)_{j_1 \cdots j_q} >$$

where for a form $\omega \in W \otimes V \otimes \bigwedge^q \underline{p}^+$, we define $\omega_{j_1,\cdots j_q} = \omega(X_{j_1}, \cdots X_{j_q})$. Note that the factor $(q-1)!$ disappears because we are taking a sum over all ordered sets of the form $(j_1, ... j_q)$. From the antisymmetry of c_{ij}^a and $[X_i, X_j]$ we get

$$\Phi_1(\eta) = -q^{-1} \sum_{\substack{i,j,a \\ j_1,\cdots j_q}} c_{ij}^a < (ad_+^q(X_a)\eta)_{j_1,\cdots j_q},\ R(X_i)R(X_j)\eta_{j_1,\cdots j_q} >$$

It can be easily checked that

$$(ad_+^q(X_a)\eta)(X_{j_1,\cdots j_q}) = -\sum_{k,u} (-1)^{u-1} c_{kj_u}^a \eta_{kj_1 \cdots \hat{j}_u \cdots j_q}$$

where the sum over u runs from 1 to q. Thus,

$$q\Phi_1(\eta) = \sum_{\substack{i,j,k,u \\ j_1,\cdots j_q}} (-1)^{u-1} c_{ij}^a c_{kj_u}^a < \eta_{kj_1 \cdots \hat{j}_u \cdots j_q},\ R(X_i)R(X_j)\eta_{j_1 \cdots j_q} >$$

Since by (3), $\sum_a c_{ij}^a c_{kj_u}^a = R_{ijkj_u}$, the sum over a can be written in terms of R_{ijkl}. So

$$q\Phi_1(\eta) = \sum_{\substack{i,j,k,u \\ j_1,\cdots j_q}} (-1)^{u-1} R_{ijkj_u} (\eta_{kj_1 \cdots \hat{j}_u \cdots j_q},\ R(X_i)R(X_j)\eta_{j_1 \cdots j_q})$$

Since R is unitary, we have

$$q\Phi_1(\eta) = -\sum_{\substack{i,j,k,u \\ j_1,\cdots j_q}} (-1)^{u-1} R_{ijkj_u} (R(X_i)\eta_{kj_1 \cdots \hat{j}_u \cdots j_q},\ R(X_j)\eta_{j_1 \cdots j_q})$$
$$= -\sum_{\substack{i,j,k,u \\ j_1,\cdots j_q}} R_{ijkj_u} (R(X_i)\eta_{kj_1 \cdots \hat{j}_u \cdots j_q},\ R(X_j)\eta_{j_u j_1 \cdots \hat{j}_u \cdots j_q})$$

Writing ℓ for j_u and rewriting the indices $(j_1, \cdots \hat{j}_u, \cdots j_q)$ as $(j_2, \cdots, j_q)$ the above expression can be written

$$q\Phi_1(\eta) = -q \sum_{\substack{i,j,k,l \\ j_2,\cdots j_q}} R_{ijkl} < R(X_i)\eta_{kj_2\cdots j_q}, R(X_j)\eta_{\ell j_2\cdots j_q} >$$

(14) Thus $\Phi_1(\eta) = -\sum_{\substack{i,j,k,l \\ j_2,\cdots j_q}} R_{ijkl} < R(X_i)\eta_{kj_2\cdots j_q}, R(X_j)\eta_{lj_2\cdots j_q} >$

3.5. We now simplify $\Phi_2(\eta)$. We have

$$\begin{aligned}\Phi_2(\eta) &= +\frac{(q-1)!}{2}\sum_a L(X_a, X_a) < ad_+^q(X_a)\eta,\ \sigma(X_a)\eta > \\ &= -\frac{(q-1)!}{2}\sum_a L(X_a, X_a) < \eta,\ ad_+^q(X_a)\sigma(X_a)\eta >\end{aligned}$$

$$\text{Since } ad_+^q(X_a)\sigma(X_a) = \frac{1}{2}[(ad_+^q \otimes \sigma)(X_a)^2 - ad_+^q(X_a)^2 \otimes 1 \\ -1 \otimes \sigma(X_a)^2]$$

we have

(15)
$$\Phi_2(\eta) = -\frac{(q-1)!}{4} < \eta,\ [(ad_+^q \otimes \sigma)(C_L) - ad_+^q(C_L) - \sigma(C_L)]\eta >$$

3.6. Substituting the expressions obtained for $\Phi_1(\eta)$ in (14), for $\Phi_2(\eta)$ in (15) and for $\Phi(\eta)$ in (10) into (13), we obtain,

$$\begin{aligned} 0 &= \frac{(q-1)!}{4} < \eta, [(ad_+^q \otimes \sigma)(C_L) + ad_+^q(C_L) - \sigma(C_L)]\eta > \\ &+ \sum_{\substack{i,j,k,l \\ j_2,\cdots j_q}} R_{ijkl} < R(X_i)\eta_{kj_2\cdots j_q},\ R(X_j)\eta_{lj_2\cdots j_q} > \end{aligned} \quad (16)$$

3.7. So far, working in analogy with the case of constant coefficients, we have obtained an expression for $\Phi(\eta)$, for η any $(0, q)$-form with values in $E(\sigma)$. *However* we have a mixed expression involving derivatives $R(X_i)\eta$ of η, together with terms involving essentially the square of the norm of η. To get rid of the dependence on the norm of η, we assume that η is an *eigenform* for the Casimir operator C_G of G. Using this we will be able to express the norm of η in terms of inner product involving the derivative $R(X_i)\eta$.

We saw in § 1, that harmonic forms are eigenforms of the Casimir operator. Let μ be the highest weight of a representation of $K^{\mathbf{C}}$ and η be a $\overline{\partial}$–harmonic form on M with values in the automorphic vector bundle $E(\sigma)$. Then by Falting's result, Theorem 2, and the Remark 2.5 following it, $R(C_G)\eta = L(C_G)\eta = \sigma(C_G)\eta$. Hence we assume from now onwards that η is an eigenform of the Casimir operator C_G of G with eigenvalue $\sigma(C_G)$ for the action R of $\underline{g}$.

In terms of the basis $\{X_a, X_i\}$ of $\underline{g}$, $C_G = -\sum X_a^2 + \sum X_i^2$. Let

$$C_k = -\sum_a X_a^2 \tag{17}$$

$$\begin{aligned}\sigma(C_G) \parallel \eta \parallel^2 &= < R(C_G)\eta, \eta > \\ &= \sum_a - < R(X_a)^2\eta, \eta > + \sum_i < R(X_i)^2\eta, \eta > \\ &= \sum_a < R(X_a)\eta, R(X_a)\eta > - \sum_i < R(X_i)\eta, R(X_i)\eta >\end{aligned}$$

The last equality follows because R is unitary.

By (6), we get,

$$\sigma(C_G) \parallel \eta \parallel^2 = < \eta,\ (ad_+^q \otimes \sigma)(C_k)\eta > - \sum_i \parallel R(X_i)\eta \parallel^2 \tag{18}$$

3.8. Write $\bigwedge^q \underline{p}^+ = \bigoplus_\alpha V(\alpha)$ as $\underline{k}^{\mathbf{C}}$- modules, where $V(\alpha)$, stands for the isotypical component of $\bigwedge^q \underline{p}^+$ corresponding to irreducible representation α of $\underline{k}^{\mathbf{C}}$. Write $V_\mu \otimes_{\mathbf{C}} \bigwedge^q \underline{p}^+ = \bigoplus_\alpha V_\mu \otimes V\alpha = \bigoplus_\beta V_\beta$, where $V(\beta)$ is the $\underline{k}^{\mathbf{C}}$– isotypical component of $V_\mu \otimes \bigwedge^q \underline{p}^+$ corresponding to the irreducible representation β, with highest weight λ. Since C_k and C_L are linear combinations of Casimir operators of the various simple components of $\underline{k}^{\mathbf{C}}$, they act as scalars on $V(\beta)$.

Write $\eta = \sum \eta_\beta$, where η_β are the projections to $W \otimes V_\beta$ of a form $\eta \in W \otimes V_\mu \otimes \bigwedge^q \underline{p}^+$. Since η is assumed to be an eigenvector of C_G with respect to the action R of $\underline{g}$ on W, the same is true of the components

η_β of η. Hence we assume from now onwards that $\eta \in W \otimes V(\beta)$, where β is a representation of $\underline{k}^{\mathbf{C}}$ occuring in $V_\mu \otimes \bigwedge^q \underline{p}^+$. Thus from (21)

$$\sigma(C_G) \parallel \eta \parallel^2 = < (ad_+^q \otimes \sigma)(C_k)\eta,\ \eta > - \sum_i \parallel R(X_i)\eta \parallel^2$$

$$(19) \qquad = \beta(C_k) \parallel \eta \parallel^2 - \sum_i \parallel R(X_i)\eta \parallel^2$$

If $\sigma(C_G) = \beta(C_k)$, then $R(X_i)\eta = 0\ \forall i$. Thus $\eta \in W^{\underline{g}} \otimes V_\beta$, where $W^{\underline{g}}$ is the invariants of $\underline{g}$ and since C_G acts on the harmonic form via the action R with eigenvalue$\sigma(C_G)$, we have $0 = \sigma(C_G) = \beta(C_k)$. Hence $\mu = 0$. Since there are no $\underline{k}^C$- trivial subspaces of $\bigwedge^q \underline{p}^+$ for $q > 0$ this says $q = 0$, and we are reduced to calculating the zeroth cohomology.

Hence we assume from now onwards that the following positivity condition holds:

$$(20) \qquad \beta(C_k) - \sigma(C_G) > 0$$

Then

$$(21) \qquad \parallel \eta \parallel^2 = \frac{\sum_i \parallel R(X_i)\eta \parallel^2}{\beta(C_k) - \sigma(C_G)}$$

3.9. We now construct a quadratic form on $\underline{p} \otimes \underline{p}$. Using the fact that η takes values in the space V_β, (16) takes the form

$$0 = \frac{(q-1)!}{4} \frac{(\beta(C_L) + \alpha(C_L) - \sigma(C_L)) \parallel \eta \parallel^2}{(\beta(C_k) - \sigma(C_G))} + \sum_{\substack{i,j,k,l \\ j_2, \cdots j_q}} R_{ijkl} < R(X_i)\eta_{kj_2\cdots j_q},\ R(X_j)\eta_{lj_2\cdots j_q} >$$

Substituting for $\parallel \eta \parallel^2$ (21), and expanding in terms of the coefficient functions $\eta_{i_1 \cdots i_q}$, we get

$$0 = \sum_{j_2 \cdots j_q} \left\{ \frac{(\beta(C_L) + \alpha(C_L) - \sigma(C_L))}{4q(\beta(C_k) - \sigma(C_G))} \sum_{i,j} \parallel R(X_i)\eta_{jj_2\cdots j_q} \parallel^2 \right.$$

$$(22) \qquad \left. + \sum_{i,j,k,l} R_{ijkl} < R(X_i)\eta_{kj_2\cdots j_q}, R(X_j)\eta_{lj_2\cdots j_q} > \right\}$$

Let

$$(23) \qquad D = D(\sigma, q, \alpha, \beta) = \frac{1}{4q} \frac{\beta(C_L) + \alpha(C_L) - \sigma(C_L)}{\beta(C_k) - \sigma(C_G)}$$

D is an explicit constant and depends only on σ, q, α and β.

Fix an orthonormal basis $e_1, \cdots e_s$ of V. Given an element v of V, let (v_r) be the co-ordinates of v with respect to this basis.

We obtain from (22),

$$0 = \sum_{j_2,\cdots,j_q} \sum_{r=1}^{s} \Bigg\{ \{D \sum_{i,j} \| (R(X_i)\eta_{jj_2\cdots j_q})_r \|^2$$

$$(24) \qquad + \sum_{i,j,k,l} R_{ijkl} < (R(X_i)\eta_{kj_2\cdots j_q})_r, (R(X_j)\eta_{lj_2\cdots j_q})_r > \Bigg\}$$

The expression inside the curly brackets, can be thought of as an equation in $\underline{p} \otimes \underline{p}$, involving the i, j variables, i.e. an expression in $R(X_i)\eta_{j*}$. Let $\xi = (\xi_{ij})_{1 \leq i,j \leq d}$ be an element of $\underline{p} \otimes \underline{p}$. Consider the following quadratic form $H = H(\underline{g}, q, \mu, \beta)$, defined on $\underline{p} \otimes \underline{p}$ as follows:

$$(25) \qquad H(\xi) = D \| \xi \|^2 + \sum_{i,j,k,l} R_{ijkl} \xi_{ik} \xi_{jl}$$

If the form H is shown to be positive definite on $\underline{p} \otimes \underline{p}$, then we see from (24), that for any $j_2, \cdots j_q$ and r,

$$(R(X_i)\eta_{j,j_2\cdots j_q})_r = 0$$

where we have assumed that η is a $(0, q)-$ form with values in V_μ, which is an eigenform of the Casimir with eigenvalue $\sigma(C_G)$. By a similar reasoning which led to the positivity condition (20), we see that $\sigma = 0$ and that the form η is invariant. We have proved the following:

Theorem 3. *Let σ be an irreducible $\underline{k}^C-$representation on V. Let β be an irreducible $\underline{k}^C$-module occuring in $V \otimes \bigwedge^q \underline{p}^+$. If the quadratic form H constructed as above is positive definite, for all β satisfying the condition 20, i.e., $\beta(C_k) - \sigma(C_G) > 0$, then*

$$H^q(M, E_\sigma) = (0)$$

unless σ is the trivial representation and $q = 0$.

Remark. We have actually proved that under the hypothesis of the theorem, nonzero eigenforms of the Casimir C_G of G with eigenvalue $\sigma(C_G)$ do not exist, unless σ is the trivial representation and $q = 0$.

Remark. We note that our conditions on the vanishing of cohomology is independent of the lattice Γ and depends only on G, q and σ. We note also that H is the sum of two quadratic forms - one the form $\xi \mapsto \sum_{i,j,k,l} R_{ijkl}\xi_{ik}\xi_{jl}$ which depends only on the geometry of G/K and not at all on the representation σ nor on the degree q, the other being the form $\xi \mapsto D \parallel \xi \parallel^2$, which depends on μ, q and β.

Remark. Our vanishing result is applicable even when the highest weight μ of V is not regular for $\underline{g}$. For example, in the next chapter we need to show vanishing of cohomology groups for certain μ vanishing on the center of $\underline{k}$. If $\underline{g}$ is simple then the center of $\underline{k}$ is generated by ρ_n. This will force $\mu = 0$, if μ is the highest weight of a representation of $\underline{g}$. Hence our results are not covered by the vanishing results of Matsushima and Murakami. It can be seen that the vanishing results we need to prove rigidity in the next chapter are not covered by the vanishing results of Hotta-Parthasarathy either. See ([15]).

3.10. We can write $\underline{p}\otimes\underline{p} = S^2\underline{p}\oplus\bigwedge^2\underline{p}$ where $S^2\underline{p}$ (resp. $\bigwedge^2\underline{p}$) is the space of symmetric (resp. skew-symmetric) 2-tensors of $\underline{p}$. These spaces are stable under $\underline{k}^{\mathbf{C}}$. Arguing as in ([17]), we see that the decomposition above is orthogonal with respect to the quadratic form H. Further if D is a positive real number, then H is positive definite when restricted to $\bigwedge^2\underline{p}$. For $\xi = (\xi_{ij}) \in S^2\underline{p}$, let $P : S^2\underline{p} \to S^2\underline{p}$ be

$$P(\xi)_{ik} = \sum_{j,l} R_{ijkl}\xi_{jl} \tag{26}$$

From the symmetry properties satisfied by R_{ijkl}, it is easy to see that P is a symmetric operator on $S^2\underline{p}$. Hence in order to show that H is positive definite on $\underline{p}\otimes\underline{p}$, it is enough to show that the form H on $S^2\underline{p}$,

$$H(\xi) = D \parallel \xi \parallel^2 + (\xi, P\xi), \quad \xi \in S^2\underline{p}$$

is positive definite on $S^2\underline{p}$.

Let λ_1 be the minimum eigenvalue of P. λ_1 depends only on G and can be explicitly calculated. For the values of λ_1, we refer to Table 1. We see that if $|\lambda_1| < D(\sigma, q, \alpha, \beta)$, then the form $H(\sigma, q, \alpha, \beta)$ is positive definite.

Corollary 1. *Assume D is positive. If $|\lambda_1| < D(\sigma, q, \alpha, \beta)$ for all β an irreducible $\underline{k}^C$-module occuring in $V \otimes \bigwedge^q \underline{p}^+$ and satisfying the positivity condition (20), then*

$$H^q(M, E_\sigma) = (0)$$

3.11. Example. When the representation σ is the trivial representation, then $\beta \simeq \alpha$. There is only one value of D to consider and $D > 0$. We have from (23),

$$D = \frac{1}{4q}\frac{2\alpha(C_L)}{\alpha(C_k)} > \frac{A}{2q}$$

The inequality $A/2q + \lambda_1 > 0$ is the one considered by Matsushima, to conclude the vanishing of Betti numbers below some degree of M. Thus from Table 1, we conclude that $A/2 + \lambda_1 \geq 0$ is true in the following cases: $I_{m_1,m_2}(m_1 \geq m_2 \geq 2)$, $II_m(m \geq 4)$, $III_m(m \geq 2)$, $IV_m(m \geq 3)$, V, VI. Hence in these cases the first Betti number of M vanishes.

3.12. We will now remove the dependence on the choice of β, when $[\underline{k}^C, \underline{k}^C]$ is simple. We will show now that when $[\underline{k}^{\mathbf{C}}, \underline{k}^{\mathbf{C}}]$ is a simple Lie algebra, then under some conditions on the highest weight μ of σ, D is positive and that the minimum value $D_{min} = \min_\beta D(\sigma, q, \alpha, \beta)$ is assumed for β with the highest weight $\lambda + \mu$.

Lemma 1. *Assume that $[\underline{k}^C, \underline{k}^C]$ is a simple Lie algebra. Let ρ_n be half the sum of positive noncompact roots. If $(\mu, \rho_n) \geq 0$, then $D > 0$.*

Proof. Write $\underline{k} = [\underline{k}^{\mathbf{C}}, \underline{k}^{\mathbf{C}}] \bigoplus \underline{k}_0$, where $\underline{k}_0$ is the one dimensional center of $\underline{k}^{\mathbf{C}}$. If τ is an irreducible representation of $\underline{k}^{\mathbf{C}}$, then τ can be written as $\tau' \oplus \tau_0$, where τ' is trivial on $\underline{k}_0$ and τ_0 is a character on $\underline{k}_0$. Let C'_L (resp. C^0_L) be the Casimir of $[\underline{k}^{\mathbf{C}}, \underline{k}^{\mathbf{C}}]$ (resp. $\underline{k}_0$) with respect to the inner product defined by L. By (7) $L \,|\, \underline{k}_0 = B \,|\, \underline{k}_0$, and by the assumption on $\underline{k}^{\mathbf{C}}$, we have $C'_L = AC'_K$, where $A = \min_a |L(X_a, X_a)|$. Hence

$$\begin{aligned} \tau(C_L) &= \| \tau_0 \|^2 + A\tau'(C'_K) \\ &= (1-A) \| \tau_0 \|^2 + A\tau(C_k) \qquad (27) \end{aligned}$$

$$\begin{aligned} \text{Hence } \operatorname{num}(D) &= \beta(C_L) - \sigma(C_L) + \alpha(C_L) \\ &= (1-A)\{\| \beta_0 \|^2 - \| \mu_0 \|^2 + \| \alpha_0 \|^2\} \\ &\quad + A\{\beta(C_k) - \sigma(C_k) + \alpha(C_k)\} \qquad (28) \end{aligned}$$

where num(D) denotes the numerator of D. Since $\beta_0 = \mu_0 \otimes \alpha_0$, we have

$$\| \beta_0 \|^2 - \| \mu_0 \|^2 = \| \alpha_0 \|^2 + 2(\mu_0, \alpha_0).$$

It is known that ρ_n is a nonzero element of $(\underline{t}^{\mathbf{C}})^*$, (see [12] (page 11, Lemma 4.2)) which is trivial on $\underline{t}^{\mathbf{C}} \cap [\underline{k}^{\mathbf{C}}, \underline{k}^{\mathbf{C}}]$. Moreover $(\alpha, \rho_n) > 0$. Since by assumption $(\mu, \rho_n) \geq 0$, we see that μ_0 is a nonnegative multiple of α_0. Hence $(\mu_0, \alpha_0) \geq 0$. Further, by the positivity condition (20),

$$\beta(C_k) - \sigma(C_G) = \beta(C_k) - \sigma(C_k) - 2(\mu, \rho_n) > 0 \tag{29}$$

Since $(\mu, \rho_n) \geq 0$, this shows that

$$\beta(C_k) - \sigma(C_k) > 0$$

Hence D is positive.

Since $\alpha(C_k)$ is independent of the irreducible $\underline{k}^{\mathbf{C}}$-constituent α of $\bigwedge^q \underline{p}^+$ ([12, Lemma 4.1]), we have that,

$$\alpha(C_L) = (1 - A) \| \alpha_0 \|^2 + A\alpha(C_k) \tag{30}$$

is independent of the irreducible constituent α of $\underline{k}^{\mathbf{C}}$ occuring in $\bigwedge^q \underline{p}^+$.

Substituting ((27), (28) and (29) into the expression for D (23), we obtain,

$$\begin{aligned} D(\sigma, q, \alpha, \beta) = {} & \frac{A}{4q} + \frac{A}{4q} \frac{2(\mu, \rho_n) + \alpha(C_k)}{\beta(C_k) - \sigma(C_k) - 2(\mu, \rho_n)} \\ & + \frac{(1 - A)}{4q} \frac{2 \| \alpha_0 \|^2 + 2(\alpha_0, \mu_0)}{\beta(C_k) - \sigma(C_k) - 2(\mu, \rho_n)} \end{aligned} \tag{31}$$

We notice that all the individual summands are positive if we assume $(\mu, \rho_n) \geq 0$.

It is known that $ad^q{}_+(C_k)$ acts by a scalar on $\bigwedge^q \underline{p}^+$ ([12] (Lemma 4.1)). Hence the value of $\alpha(C_k)$ is independent of the irreducible constituent α occuring in $ad^q{}_+$. Morever the dependence on β occurs in the denominators of the summands. Hence the minimal positive value of D occurs when $\beta(C_k)$ is maximum amongst the representations β occuring in $V(\sigma) \otimes \bigwedge^q \underline{p}^+$. This happens when the highest weight of

$\beta = \mu + \alpha$. Substituting for β we obtain the minimal value $D_{\min}$ of D as

$$D_{\min} = \frac{A}{4q} + \frac{A}{4q}\frac{2(\mu,\rho_n)+\alpha(C_k)}{\alpha(C_k)+2(\mu,\alpha-\rho_n)} + \frac{(1-A)}{2q}\frac{\|\alpha_0\|^2+(\alpha_0,\mu_0)}{\alpha(C_k)+2(\mu,\alpha-\rho_n)} \tag{32}$$

To obtain vanishing results one has to show by Corollary (1), that $D_{\min} > |\lambda_1|$, where λ_1 is the minimal eigenvalue of P. Hence we have

Theorem 4. *Assume that $[\underline{k}^C, \underline{k}^C]$ is a simple Lie algebra. Let σ be an irreducible representation of $\underline{k}^C$ with highest weight μ. Let ρ_n be half the sum of the positive noncompact roots. Assume that $(\mu,\rho_n) \geq 0$. If*

$$D_{min} = \frac{A}{4q} + \frac{A}{4q}\frac{2(\mu,\rho_n)+\alpha(C_k)}{\alpha(C_k)+2(\mu,\alpha-\rho_n)} + \frac{(1-A)}{2q}\frac{\|\alpha_0\|^2+(\alpha_0,\mu_0)}{\alpha(C_k)+2(\mu,\alpha-\rho_n)} > |\lambda_1|$$

then the cohomology groups, $H^q(M, E(\sigma))$ vanish.

Since $D_{\min} > A/4q$, we have

Corollary 2. *Assume that $[\underline{k}^C, \underline{k}^C]$ is a simple Lie algebra. Let σ be an irreducible representation of $\underline{k}^C$ with highest weight μ. Let ρ_n be half the sum of the positive noncompact roots. Assume that $(\mu,\rho_n) \geq 0$. If*

$$\frac{A}{4q} \geq |\lambda_1|$$

then the cohomology groups, $H^q(M, E(\sigma))$ vanish.

We tabulate the constants A and λ_1 in Table 1. See ([10]) for the table.

Table 1

Type of M	A	λ_1
$I_{m_1,m_2}(m_1 \geq m_2 \geq 1)$	$m_2/(m_1+m_2)$	$-1/(m_1+m_2)$
$II_m(m \geq 3)$	$(m-2)/2(m-1)$	$-1/2(m-1)$
$III_m(m \geq 2)$	$(m+2)/2(m+1)$	$-1/(m+1)$
$IV_m(m \geq 3)$	$2/m$	$-1/m$
V	$1/3$	$-1/12$
VI	$1/3$	$-1/18$

Note. For I_{m_1,m_2}, the other value is $A_2 = m_1/(m_1 + m_2)$.

From Table 1 we obtain for the first cohomology,

Corollary 3. *Let G be any one of the following type: $II_m(m \geq 6), III_m(m \geq 6)$ and the two exceptional types V, VI. Let σ be an irreducible representation of $\underline{k}^C$, with highest weight μ such that $(\mu, \rho_n) \geq 0$. Then*

$$H^1(M, E(\sigma)) = (0)$$

4. Application to deformations

4.1. In this section we look at deformations of complex structures on locally homogeneous Kahlerian manifolds. We use the notation of the previous sections. Let K' be a connected, closed subgroup of G. From now onwards we assume that there is a G-invariant Kähler structure on G/K'. Then the following is known: ([1])

K' is compact and is the centralizer of a torus S in G. Let T be a maximal compact torus in G containing S. Then T is a Cartan subgroup in G. Let K be the maximal compact subgroup of G containing T. Then G/K is a hermitian symmetric domain and the natural map $\widetilde{\pi} : G/K' \to G/K$ is holomorphic.

Let $\underline{k}'$, $\underline{k}'^{\mathbf{C}}$, K', $K'^{\mathbf{C}}$ denote the usual objects associated with K'. Let P_0 be the Borel subgroup of $G^{\mathbf{C}}$, whose Lie algebra is spanned by the negative root spaces corresponding to the ordering chosen above. Let Q' denote the parabolic subgroup $K'^{\mathbf{C}}P_0$, with Levi component $K'^{\mathbf{C}}$. The complex strucures on G/K' is defined in a manner analogous to that of G/K : $G/K' \subset G^{\mathbf{C}}/Q'$ is open and the complex structure on G/K' is the induced one. Since $Q' \subset Q$, the projection map $\widetilde{\pi}$: $G/K' \to G/K$ is holomorphic.

Conversely given a parabolic subgroup Q' of G contained in Q, let $K' = P' \cap K$. Then $K'^{\mathbf{C}}$ is a Levi component of P', and the projection map $G/K' \to G/K$ is holomorphic. Furthermore G/K' carries a G-invariant complex Kähler strucure.

Remark. Let G be a real semisimle Lie group without compact factors and let K' be a compact, connected subgroup of G. Assume that G/K' carries a G-invariant complex structure. Such spaces and their quotients by torsion-free lattices were studied by Griffiths and Schmid ([6]). These spaces arise as parametrizing spaces for variations of polarized Hodge structures. Borel's theorem says that G/K' supports a G-invariant Kahler structure iff it is fibered over hermitian symmetric domain. This allows us to apply the Leray spectral sequence to compute the cohomologies of the vector bundles we consider on $\Gamma\backslash G/K'$. The problem then reduces to computing the cohomologies of certain automorphic vector bundles on $\Gamma\backslash G/K$ to which we can apply the vanishing theorems of Section 3.

4.2. Let Γ be an irreducible, torsion-free, cocompact lattice in G. Let $M' = \Gamma\backslash G/K'$ and $M = \Gamma\backslash G/K$. Let $\pi : M' \to M$ denote the natural map. We are interested in the deformations of the complex structure on M'. Similar to the construction of automorphic vector bundles on M, given a holomorphic representation σ' of Q', one can construct a holomorphic vector bundle $E_{\sigma'}$ on M'. For a sheaf E on M (or M'), denote by $\widetilde{E}$, the pullback sheaf on G/K (resp. G/K'). Let $h^i(\sigma')$ denote the action of Q on $H^i(Q/Q', \widetilde{E}_{\sigma'} \mid (Q/Q'))$.

Proposition 5. *With notation as above,* $R^i\pi_*E_{\sigma'} \simeq E_{h^i(\sigma')}$.

Proof. Let $\widetilde{\pi} : G/K' \to G/K$ also denote the projection map. For a G-sheaf $\widetilde{E}$ on G/K', denote by $\widetilde{E}^\Gamma$, the sheaf on M' obtained by taking Γ-invariant sections of Γ invariant open sets in G/K' . Then,

$$\pi_*\left(\widetilde{E}^\Gamma_{\sigma'}\right) \simeq \left(\widetilde{\pi}_*\widetilde{E}_{\sigma'}\right)^\Gamma.$$

Since $\widetilde{E} \mapsto \widetilde{E}^\Gamma$ is an exact functor, one has $R^q\pi_*E_{\sigma'} = R^q\pi_*\left(\widetilde{E}^\Gamma_{\sigma'}\right) \simeq \left(R^i\widetilde{\pi}_*\widetilde{E}_{\sigma'}\right)^\Gamma$.

To calculate $R^i\widetilde{\pi}_*\widetilde{E}_{\sigma'}$, since $\widetilde{E}_{\sigma'}$ are restrictions of $G^{\mathbf{C}}$-sheaves on $G^{\mathbf{C}}/Q'$, it is enough to calculate $R^i\widetilde{\pi}_*\widetilde{E}_{\sigma'}$ for the map $\widetilde{\pi} : G^{\mathbf{C}}/Q' \to G^{\mathbf{C}}/Q$. Since $R^i\widetilde{\pi}_*\widetilde{E}_{\sigma'}$ is again a $G^{\mathbf{C}}$-sheaf on $G^{\mathbf{C}}/Q$, it is the sheaf induced from the representation of Q, on the fiber of $R^i\widetilde{\pi}_*\widetilde{E}_{\sigma'}$. Since $\widetilde{\pi} : G^{\mathbf{C}}/Q' \to G^{\mathbf{C}}/Q$ is a locally trivial fibration, this representation is just the action of Q on $H^i(Q/Q', \widetilde{E}_{\sigma'} \mid_{(Q/Q')})$, which is $h^i(\sigma')$. Hence the result.

4.3. Let $\Theta_{M'}$ denote the sheaf of germs of holomorphic vector fields on M'. We now calculate the cohomology of $\Theta_{M'}$. On M' there is a short exact sequence of sheaves,

$$0 \longrightarrow \Theta_{M'/M} \longrightarrow \Theta_{M'} \longrightarrow \pi^*\Theta_M \longrightarrow 0 \tag{33}$$

where Θ_M is the sheaf of germs of holomorphic vector fields on M and $\Theta_{M'/M}$ is the sheaf of germs of holomorphic vector fields which are tangential to the fibers of the map $M' \longrightarrow M$.

By projection formula

$$R^i\pi_*(\pi^*\Theta_M) \simeq (R^i\pi_*\mathcal{O}'_M) \otimes \Theta_M$$

By Proposition 5, $R^i\pi_*\mathcal{O}'_M$ is the sheaf associated to the representation of $K^{\mathbf{C}}$ on $H^i(K^{\mathbf{C}}/K^{\mathbf{C}} \cap Q', \mathcal{O})$. It is well known that $H^i(K^{\mathbf{C}}/K^{\mathbf{C}} \cap Q', \mathcal{O}) = (0)$ for $i \geq 0$. See ([3]). Hence

$$R^i\pi_*(\pi^*\Theta_M) = (0) \ (i > 0) \tag{34}$$

$$\pi_*(\pi^*\Theta_M) = \Theta_M \tag{35}$$

Similarly, $R^i\pi_*\Theta_{M'/M}$ is the sheaf associated to the representation of $K^{\mathbf{C}}$ on $H^p(K^{\mathbf{C}}/K^{\mathbf{C}} \cap Q', \Theta_{K^{\mathbf{C}}/K^{\mathbf{C}} \cap Q'})$. By Bott's Theorem ([3]), we have

$$H^i(K^{\mathbf{C}}/K^{\mathbf{C}} \cap Q', \Theta_{K^{\mathbf{C}}/K^{\mathbf{C}} \cap Q'}) = (0) \ (i \geq 1)$$

Hence

$$R^i\pi_*\Theta_{M'/M} = (0) \ (i > 0) \tag{36}$$

Let $T_{K/K'}$ denote the $K^{\mathbf{C}}$-module $H^0(K^{\mathbf{C}}/K^{\mathbf{C}} \cap Q', \Theta_{K^{\mathbf{C}}/K^{\mathbf{C}} \cap Q'})$, which is isomorphic to $\pi_*\Theta_{M'/M}$. The long exact sequence of direct image sheaves under π corresponding to the short exact sequence (33), reduces by (34), (35), (36) to the following short exact sequence on M:

$$0 \to E(T_{K/K'}) \to \pi_*\Theta_{M'} \to \Theta_M \to 0 \tag{37}$$

Also from equations (34) and (36), we get

$$R^i\pi_*\Theta_{M'} = (0) \ (i > 0)$$

Substituting the vanishing of $R^q\pi_*(\Theta_{M'})$ $(q > 0)$, in the Leray spectral sequence for the fibration $M' \longrightarrow M$, calculating the cohomology of $\Theta_{M'}$, we have

$$E_2^{pq} = H^p(M, R^q\pi_*\Theta_{M'}) = (0) \ (q > 0)$$

and so

$$H^p(M', \Theta_{M'}) = E_2^{p0} = H^p(M, \pi_*\Theta_{M'})$$

By the results of Calabi and Vesentini, we have

$$H^0(M, \Theta_M) = (0) \tag{38}$$

Further, since M is not a compact Riemann surface,

$$H^1(M, \Theta_M) = (0) \tag{39}$$

([4] (Theorem 1 and Corollary to Theorem 1). Inserting the information from (38) and (39) into the long exact sequence of cohomology groups corresponding to the short exact sequence (37), we see that

$$H^1(M', \Theta_{M'}) \simeq H^1(M, E(T_{K/K'}))$$

Let now $K' = T$. $\Theta_{K^{\mathbf{C}}/K^{\mathbf{C}} \cap P_0}$ is the homogeneous vector bundle on $K^{\mathbf{C}}/K^{\mathbf{C}} \cap P_0$ associated to the representation of T on $\underline{k}^{\mathbf{C}}/(\underline{k}^{\mathbf{C}} \cap \underline{p}_0)$, where $\underline{p}_0$ is the Lie algebra of P_0. By Borel-Weil-Bott Theorem,

$$T_{K/K'} = H^0(K^{\mathbf{C}}/K^{\mathbf{C}} \cap P_0, \Theta_{K^{\mathbf{C}}/K^{\mathbf{C}} \cap P_0}) \simeq [\underline{k}^{\mathbf{C}},\ \underline{k}^{\mathbf{C}}]$$

as $K^{\mathbf{C}}$-modules.

For a general $K' \supset T$, $E(T_{K/K'})$ is a direct summand of $E(T_{K/T})$ as a $\underline{k}^{\mathbf{C}}$-module. Hence if one shows that $H^1(N, \Theta_N) = (0)$, when $N = \Gamma \backslash G/T$, then $H^1(M', \Theta_{M'}) = (0)$ for $M' = \Gamma \backslash G/K'$, $K \supset K' \supset T$.

Hence we have,

Theorem 6. *With notations as above, if*

$$H^1(M, E([\underline{k}^C, \underline{k}^C]) = (0)$$

then the complex structure on any M' is infinitesimally rigid.

4.4. The fibration $M' \to M$ can also be thought of as a $K^{\mathbf{C}}/Q'^{\mathbf{C}}$ bundle associated to the $K^{\mathbf{C}}$ principal bundle on M. $H^1(M, E(T_{K/K'}))$ parametrizes the space of infinitesimal deformations of the bundle $M' \to M$. The Kuranishi space of deformations of the bundle $M' \to M$ thus has tangent space $H^1(M, E(T_{K/K'})) \simeq H^1(E([\underline{k}^{\mathbf{C}}, \underline{k}^{\mathbf{C}}]))$. By what has been said before Theorem 6, $H^1(M, E(T_{K/K'})) \simeq H^1(M', \Theta)$. Morever it is clear that there is a natural map from the Kuranishi space corresponding to the deformations of the bundle $M' \to M$, to the Kuranishi space corresponding to the deformations of the complex structure on M'. There is actually a map from the differential graded Lie algebra corresponding to the deformations of the bundle $M' \to M$ to the differential graded Lie algebra corresponding to the deformations of the complex structure on M'. This map is an isomorphism at the first cohomology level, i.e., at the level of Zariski tangent spaces of the Kuranishi spaces corresponding to the deformations. Morever from the degeneration of the Leray spectral sequence at the E_2 stage and the vanishing of $H^1(M, \Theta_M)$ by Calabi-Vesentini, it follows that there is a natural inclusion of $H^2(M, E(T_{K/K''}))$ into $H^2(M', \Theta_{M'})$. It follows from general facts on differential graded Lie algebras that the corresponding Kuranishi spaces are actually isomorphic. See ([13] Comparison theorem)).

Theorem 7. *Under the natural map, the Kuranishi space of deformations of the bundle $M' \to M$, is isomorphic to the Kuranishi space of deformations of the complex structure on M'.*

4.5. Write $\underline{k}^{\mathbf{C}} = \underline{k}_0 \oplus \cdots \oplus \underline{k}_r$ where $\underline{k}_0$ is the center of $\underline{k}^{\mathbf{C}}$ and $\underline{k}_1, \cdots \underline{k}_r$ are the simple ideals of $\underline{k}^{\mathbf{C}}$. Thus in order to show the rigidity of the complex structure on spaces of the form $\Gamma\backslash G/T$, it is enough to show that $H^1(M, E(\underline{k}_i))$ vanishes for $i = 1, \cdots r$, where $M = \Gamma\backslash G/K$ and $E(\underline{k}_i)$ is the automorphic vector bundle on M associated to the representation of K on $\underline{k}_i$. We now calculate the constants D. Let σ denote any of the representations of $\underline{k}^{\mathbf{C}}$ on a simple component $\underline{k}_i$ of $\underline{k}^{\mathbf{C}}$. σ is trivial on the center $\underline{k}_0$ of $\underline{k}^{\mathbf{C}}$. Since ρ_n is trivial on $[\underline{k}^{\mathbf{C}}, \underline{k}^{\mathbf{C}}]$, we obtain that $(\mu, \rho_n) = 0$ where μ is the highest weight of σ. Hence

$$\sigma(C_G) = \| \mu \|^2 + 2(\mu, \rho_k + \rho_n) = \| \mu \|^2 + 2(\mu, \rho_k) = \sigma(C_k) \tag{40}$$

We are interested in the first cohomology of $E(\sigma)$. We assume now that $\underline{g}^{\mathbf{C}}$ is simple. Then the representation $\alpha = ad^1_+$ of $\underline{k}^{\mathbf{C}}$ on $\underline{p}^+$ is irreducible. τ is then an irreducible $\underline{k}^{\mathbf{C}}$- constituent of $\underline{k}_i \otimes \underline{p}^+$. We will do the computations when $[\underline{k}^{\mathbf{C}}, \underline{k}^{\mathbf{C}}]$ is simple and the case when G is a group of the type $I_{m_1,m_2} (m_1 \geq m_2 \geq 2)$.

4.6. In this section we assume that $[\underline{k}^{\mathbf{C}}, \underline{k}^{\mathbf{C}}]$ is simple. Since σ is trivial on the center $\underline{k}_0$, we have by (32) and (40),

$$D_{\min} = \frac{A}{4} + \frac{1}{4} \frac{2(1-A) \| \alpha_0 \|^2 + A\alpha(C_k)}{\alpha(C_k) + 2(\mu, \alpha)} \tag{41}$$

Note. For the notation concerning root systems, we follow [8] (Chapter X, Section3). We refer to Table 1, for the values of A and λ_1. First of all, for the exceptional groups of type V, VI we see from Table 1, that

$$D + \lambda_1 > \frac{A}{4} + \lambda_1 \geq 0 \tag{42}$$

4.6.1. We now consider groups of type II_m. $II_m (m \geq 3) : G = SO^*(2m), K = U(m)$
Roots: $\pm e_i \pm e_j$ $(1 \leq i \neq j \leq m)$
Compact roots: $\pm(e_i - e_j)$ $(1 \leq i \neq j \leq m)$ Non compact roots: $\pm(e_i + e_j)$ $(1 \leq i \neq j \leq m)$

$$2\rho_k = (m-1)e_1 + (m-3)e_2 + \cdots - (m-1)e_m$$

$$\mu = e_1 - e_m, \quad \alpha = e_1 + e_2$$

$$\| \alpha \|^2 = 4, \quad (\alpha, \mu) = 1, \quad (\alpha, 2\rho_k) = 2m - 4, \quad \alpha(C_k) = 2m$$

$$\underline{k}_0 = \mathbf{C}(1, \cdots, n), \quad \alpha_0 = \frac{2}{m}(1, \cdots, 1), \quad \| \alpha_0 \|^2 = \frac{4}{m}$$

$$D_{\min} + \lambda_1 = \frac{2m^2 - 7m - 2}{8(m-1)(m+1)}$$

Hence

$$D + \lambda_1 > 0 \ \text{ if } m \geq 4 \tag{43}$$

4.6.2. We now consider groups of type III_m. $III_m (m \geq 2) : G = Sp(2m, \mathbf{R}), K = U(m)$
Roots: $\pm 2e_i$, $\pm(e_i \pm e_j)$ $(1 \leq i \neq j \leq m)$
Compact roots: $\pm(e_i - e_j)$ $(1 \leq i \neq j \leq m)$
Noncompact roots: $\pm 2e_i, \pm(e_i + e_j)$ $(1 \leq i \neq j \leq m)$

$$2\rho_k = (m-1)e_1 + \cdots - (m-1)e_m$$

$$\mu = e_1 - e_m, \quad \alpha = 2e_1$$

$$\underline{k}_0 = \mathbf{C}(e_1, \cdots, e_m), \quad \alpha_0 = \frac{2}{m}(1, \cdots, 1), \quad \| \alpha_0 \|^2 = \frac{4}{m}$$

$$(\alpha, \mu) = 2, \quad (2\rho_k, \alpha) = 2(m-1), \quad \| \alpha \|^2 = 4, \quad \alpha(C_k) = 2m + 2$$

$$D_{\min} + \lambda_1 = \frac{m^2 - 6}{4(m+1)(m+2)}$$

Thus

$$D + \lambda_1 > 0 \quad \text{if} \quad m \geq 3 \tag{44}$$

4.6.3. We now consider groups of type IV_m, for m even. IV_m : $(m+2 = 2l,\ m \geq 3, m \neq 4)$ $G \simeq SO(m,2)$, $K \simeq SO(m)$
Roots: $\pm e_i \pm e_j$ $(1 \leq i \neq j \leq \ell)$
Compact roots: $\pm e_i \pm e_j$ $(2 \leq i \neq j \leq l)$
Noncompact roots: $\pm e_1 \pm e_j$ $(2 \leq j \leq \ell)$

$$2\rho_k = 2(l-1)e_2 + 2(\ell - 4)e_3 + \cdots$$

$$\mu = e_2 + e_3, \quad \alpha = e_1 + e_2$$

$$\underline{k}_0 = \mathbf{C}e_1, \quad \alpha_0 = e_1, \quad \| \alpha_0 \|^2 = 1$$

$$(\alpha, \mu) = 1, \quad \| \alpha \|^2 = 2, \quad (2\rho_k, \alpha) = 4\ell - 10, \quad \alpha(C_k) = 4\ell - 8 = 2m - 4$$

$$D_{\min} + \lambda_1 = \frac{m-4}{4m(m-1)}$$

Thus

$$D + \lambda_1 > 0 \quad \text{if} \quad m > 4 \tag{45}$$

4.6.4. We now consider groups of type IV_m, for m odd. IV_m : $(m+1 = 2l,)$ $G \simeq SO(m,2)$, $K \simeq SO(m)$
Roots: $\pm e_i$, $\pm e_i \pm e_j$ $(1 \leq i \neq j \leq \ell)$
Compact roots: $\pm e_i$, $\pm e_i \pm e_j$ $(2 \leq i \neq j \leq \ell)$
Noncompact roots: $\pm e_1$, $\pm e_1 \pm e_j$ $(2 \leq j \leq \ell)$

$$2\rho_k = (2\ell - 1)e_1 + (2\ell - 3)e_2 + \cdots$$

$$\mu = e_2 + e_3, \quad \alpha = e_1 + e_2$$

$$\underline{k}_0 = \mathbf{C}e_1, \quad \alpha_0 = e_1, \quad \| \alpha_0 \|^2 = 1$$

$$(\alpha, \mu) = 1, \quad \| \alpha \|^2 = 2, \quad (2\rho_k, \alpha) = 4\ell - 4, \quad \alpha(C_k) = 4\ell - 2 = 2m$$

$$D_{\min} + \lambda_1 = \frac{m-4}{4m(m+1)}$$

Thus

$$D + \lambda_1 > 0 \quad \text{if} \, m > 4 \tag{46}$$

4.7. We now consider groups in Type I.

When $G \simeq SU(n,1)$, a similar calculation done as above will show that $D_{\min} + \lambda_1$ is negative. Hence in this case we cannot conclude anything about rigidity of the complex structure on $\Gamma\backslash G/T$.

4.7.1. We now consider the case when $\underline{g} \simeq su(m_1, m_2)$ $(m_1 \geq m_2 \geq 2)$ Let $m = m_1 + m_2$. We identify $\underline{g}^{\mathbf{C}} \simeq s\ell(m, \mathbf{C})$. $\underline{k}^{\mathbf{C}}$ can be identified with the trace 0 matrices in $g\ell(m_1, \mathbf{C}) \times gl(m_2, \mathbf{C})$. $\underline{t}^{\mathbf{C}}$ will be idenfitied with the trace 0 vectors in $\mathbf{C}^m$. For B we will consider the usual inner product on $\mathbf{C}^m$. The inner product given by L on $\underline{t}^{\mathbf{C}}$, will be extended to $\mathbf{C}^m$ by taking the orthogonal sum with the usual inner product on scalars. B and L have the property, that the scalar matrices are orthogonal to $\underline{k}^{\mathbf{C}}$. Hence if λ is the highest weight of a representation of $g\ell(m_1) \times g\ell(m_2)$, which vanishes on the space of scalar matrices, then the values of the Casimir of $\underline{k}^{\mathbf{C}}$ and that of $g\ell(m_1, \mathbf{C}) \times g\ell(m_2, \mathbf{C})$ taken with respect to λ coincide. Moreover $\mathbf{C}^{m_1}$ and $\mathbf{C}^{m_2}$ are orthogonal with respect to either B or L. Hence the Casimir of $\underline{g\ell}(m_1) \times g\ell(m_2)$ taken with respect to either B or L, decomposes as the sum of the Casimirs of $g\ell(m_1)$ and $g\ell(m_2)$. Let C_K^i (resp. C_L^i)$(i = 1, 2)$ denote the Casimir of $g\ell(m_i)$ taken with respect to $B \mid g\ell(m_i)$. (resp. $L \mid g\ell(m_i)$). Then for a highest weight λ of a representation of $g\ell(m_1, \mathbf{C}) \times g\ell(m_2, \mathbf{C})$ vanishing on the scalars, we have

$$\lambda(C_k) = \lambda_1(C_K^1) + \lambda_2(C_K^2)$$

and

$$\lambda(C_L) = \lambda_1(C_L') + \lambda_2(C_L^2)$$

where λ_1 (resp. λ_2) denotes the projection of λ to $\mathbf{C}^m$ (resp. $\mathbf{C}^{m_2}$) .

With respect to the standard notation, the roots are:

Roots of $s\ell(m, \mathbf{C}) : \pm(e_i - e_j)(1 \leq i < j \leq m)$

Roots of $s\ell(m_1, \mathbf{C}) : \pm(e_i - e_j)(1 \leq i < j \leq m_1)$

Roots of $s\ell(m_2, \mathbf{C}) : \pm(e_i - e_j)(m_1 + 1 \leq i < j \leq m)$.

Write $\underline{k}^{\mathbf{C}} \simeq \underline{k}_0 \oplus s\ell(m_1, \mathbf{C}) \oplus s\ell(m_2, \mathbf{C})$ where $\underline{k}_0$ is the center of $\underline{k}^{\mathbf{C}}$

Let σ_i be the representation of $\underline{k}^{\mathbf{C}}$ on $s\ell(m_i, \mathbf{C})$. We have to show that $H^1(M, E(\sigma_i)) = (0)$.

The representation $\alpha = ad_+^1$ of $\underline{k}^{\mathbf{C}}$ on $\underline{p}^+$ can be identified with the representation of $g\ell(m_1) \times g\ell(m_2)$ on $\mathbf{C}^{m_1} \otimes (\mathbf{C}^{m_2})^*$, which is trivial on the scalars. The highest weight of $\alpha = e_1 - e_m$, and it vanishes on the scalars.

σ_1 (resp. σ_2) restricts to the adjoint representation of $s\ell(m_1, \mathbf{C})$ (resp. $s\ell(m_2, \mathbf{C})$) and is trivial on $s\ell(m_2, \mathbf{C})$ (resp. $s\ell(m_1, \mathbf{C})$). Hence in order to find the representations occuring in $\sigma_i \otimes \alpha (i = 1, 2)$, it is enough to decompose the tensor product representation $Ad_0 \otimes \omega_1$ of $g\ell(n, \mathbf{C})$ (for $n = m_1$ or m_2), where Ad_0 denotes the adjoint representation of $g\ell(n, \mathbf{C})$ on the space of trace 0 matrices and ω_1 is the standard representation of $g\ell(n, \mathbf{C})$ on $\mathbf{C}^n$. Note that α restricted to $gl(m_1)$ (resp. $gl(m_2)$) is isomorphic to ω_1(resp. ω_1^*).

Now $Ad_0 \subseteq Ad \simeq \mathbf{C}^n \otimes (\mathbf{C}^n)^*$ and

$$(\mathbf{C}^n \otimes (\mathbf{C}^n)^*) \otimes \mathbf{C}^n \simeq (\mathbf{C}^n \otimes \mathbf{C}^n) \otimes (\mathbf{C}^n)^*$$

$$\simeq (S^2\mathbf{C}^n) \otimes (\mathbf{C}^n)^* \oplus (\Lambda^2\mathbf{C}^n) \otimes (\mathbf{C}^n)^*$$

From the Weyl dimension formula, it is easy to see that as $\underline{g}\ell(n)$ modules,

$$(S^2\mathbf{C}^n) \otimes (\mathbf{C}^n)^* \simeq V(2e_1 - e_n) \otimes V(e_1)$$

$$\Lambda^2\mathbf{C}^n \otimes (\mathbf{C}^n)^* \simeq V(e_1 + e_2 - e_n) \otimes Ve_1)$$

where $V(2e_1 - e_n)$ (resp. $V(e_1 + e_2 - e_n)$) is the representation of $g\ell(n, \mathbf{C})$ with highest weight $2e_1 - e_n$ (resp. $e_1 + e_2 - e_n$).

Since $L \mid s\ell(m_1) = A_1 B \mid s\ell(m_1)$ and $L \mid s\ell(m_2) = A_2 B \mid s\ell(m_2)$, and L and B are same on the orthogonal complement of $s\ell(m_1) \times s\ell(m_2)$ inside $g\ell(m_1) \times g\ell(m_2)$, we have for $i = 1, 2$

$$\lambda_i(C_L^i) = (1 - A_i) \parallel \lambda_i 0 \parallel^2 + A_i \lambda_i (C_K^i)$$

where λ_{io} is the orthogonal projection of λ_i to scalar vectors in $\mathbf{C}^{m_i} (1 = 1, 2)$.

Let C be the Casimir of $g\ell(n, \mathbf{C})$ with respect to the usual inner product on $\mathbf{C}^n$. Then it is easy to see $\omega_1(C) = n$, $Ad_0(C) = 2n$. Let τ be a representation of $g\ell(n, \mathbf{C})$ occuring in $Ad_0 \otimes \omega_1$. We have the following table.

Table 2

τ	$\tau(C)$	$\tau(C) + \omega_1(C) - Ad_0(C)$
$\omega_1 \simeq V(\ell_1)$	n	0
$V(\ell_1 + \ell_2 - e_n)$	$3n - 2$	$2n - 2$
$V(2\ell_1 + \ell_2)$	$3n + 2$	$2n + 2$

4.7.2. We now calculate D. Let τ_i be a represent occuring in $\sigma_i \otimes \alpha$ having highest weight β_i.

$$
\begin{aligned}
(47)\ & D(\sigma_i, \alpha, 1, \tau_i) \\
&= \frac{1}{4} \frac{\tau_i(C_L) + \alpha(C_L) - \sigma_i(C_L)}{(\tau_i(C_k) - \sigma_i(C_k))} \\
&\simeq \frac{1}{4} \frac{\{\tau_{i1}(C_L^1) - \sigma_{iL}(C_L^1) + \alpha(C_L^1)\} + \{\tau_{i2}(C_L^2) + \alpha_2(C_L^2) - \sigma_{i2}(C_L^2)\}}{\tau_i(C_k) - \sigma_i(C_k)} \\
&\simeq \frac{1}{4} \frac{2(1 - A_1) \parallel \alpha_{10} \parallel^2 + A_1\{\tau_{i1}(C_K^1) - \sigma_{i1}(C_K^1) + \alpha_1(C_K^1)\}}{\tau_i(C_k) - \sigma_i(C_k)} \\
&\quad + \frac{1}{4} \frac{2(1 - A_2) \parallel \alpha_{20} \parallel^2 + A_2\{\tau_{i2}(C_K^2) - \sigma_{i2}(C_K^2) + \alpha_2(C_K^2)\}}{\tau_i(C_k) - \sigma_i(C_k)}
\end{aligned}
$$

From the table given above, we note that the quantities inside $\{,\}$ are ≥ 0. Since we have the condition that $\tau_i(C_k) - \sigma_i(C_k) > 0$, we see that $D > 0$. Thus in order to conclude about the vanishing of $H^1(E(\sigma_i))$ we have to check that $D + \lambda_1 > 0$ for all allowed choice 8 of τ_i. We have

$$(A_1 = \frac{m_2}{m}) \leq (A_2 = \frac{m_1}{m}).$$

Thus

$$
\begin{aligned}
D &> \frac{A_1}{4} \frac{\tau_i(C_k) - \sigma_i(C_k) + \alpha(C_k)}{\tau_i(C_k) - \sigma_i(C_k)} \\
&= \frac{A_1}{4} + \frac{A_1}{4} \frac{\alpha(C_k)}{\tau_i(C_k) - \sigma_i(C_k)} \qquad (48)
\end{aligned}
$$

Hence

$$D + \lambda_1 > \frac{A_1}{4} + \lambda_1 = \frac{m_2 - 4}{4m}$$

Hence

$$D + \lambda_1 > 0 \quad \text{if } m_1 \geq m_2 \geq 4 \qquad (49)$$

4.7.3. From (48), the minimum value of the right hand side occurs when $\tau_i(C_k)$ is the maximal possible value. Let us assume for example that $i = 1$, and $\tau_1(C_k)$ is maximum. We have

$$\tau_1(C_k) - \sigma_1(C_k) = \{\tau_{11}(C_K^1) - \sigma_{11}(C_K^1)\} + \{\tau_{12}(C_K^2) - \sigma_{12}(C_K^2)\}$$

Now σ_{12} is the trivial representation and so $\tau_{12} \simeq \alpha_2$. For τ_{11} we can assume that it has highest weight $2e_1 - e_{m_1}$. Thus in this case,

$$\tau_1(C_k) - \sigma_1(C_k) = (3m_1 + 2) - 2m_1 + m_2 = m + 2$$

Similarly when $i = 2$, we find that

$$\tau_2(C_k) - \sigma_2(C_k) = m + 2$$

We have $\alpha(C_k) = m_1 + m_2 = m$. Thus

$$\begin{aligned} D + \lambda_1 &> \frac{m_2 - 4}{4m} + \frac{m_2}{4m}\frac{m}{(m+2)} \\ &> 0 \quad \text{if } m_2 \geq 3 \end{aligned} \tag{50}$$

4.7.4. Hence we are now left with $SU(m_1, 2)$. We have that the highest weight of α_1 is e_1 and of α_2 is - e_m. Thus

$$\| \alpha_{10} \|^2 = \frac{1}{m_1}, \quad \| \alpha_{20} \|^2 = \frac{1}{m_2}.$$

From (48), we see that

$$D \geq \frac{A_1}{4} + \frac{A_1}{4}\frac{\alpha(C_k)}{\tau_i(C_k) - \sigma_i(C_k)} + \frac{1}{4}\frac{2(1 - A_1) \| \alpha_{10} \|^2 + 2(1 - A_2) \| \alpha_{20} \|^2}{\tau_i(C_k) - \sigma_i(C_k)}$$

and the equality is strict unless either $A_1 = A_2$ or $\tau_{i2}(C_K^2) - \sigma_{i2}(C_K^2) + \alpha_2(C_K^2) = 0$.

This can happen only when either $m_1 = m_2 = 2$, or that $i = 2$, and τ_{22} is the representation with Casimir acting by the scalar m_2.

Suppose $i = 2$ and $\tau_{22}(C_K^2) = m_2$.

Since σ_{21} is the trivial representation we have,

$$\tau_i(C_k) - \sigma_2(C_k) = m_1 + (m_2 - 2m_2) = m_1 - m_2$$

Since by (20), $\tau(C_k) - \sigma_i(C_k) > 0$, we have that $m_1 > m_2$. Further

$$D > \frac{A_1}{4} + \frac{A_1}{4}\frac{m}{m_1 - m_2}$$

and

$$D + \lambda_1 > \frac{(m_2 - 4)(m_1 - m_2) + m_2(m_1 + m_2)}{4m(m_1 - m_2)}$$

which is clearly positive for $m_2 \geq 2$. Thus in this case we have $D+\lambda_1 > 0$.

Coming back to the original inequality (48), on substituting for α_{i0} and taking for $\tau_i(C_k)$ the maximum possible value we find that

$$D+\lambda_1 \geq \frac{(m_2-4)(m+2)+m_2m+4}{4m(m+2)}$$

with strict inequality if $m_1 \neq 2$. When $m_2 = 2$, we see then that

$$D+\lambda_1 > 0 \quad (m_1 > m_2 \geq 2) \tag{51}$$

When $m_1 = m_2 = 2$, we get that $D+\lambda_1 = 0$. Thus except when $G \simeq SU(2,2)$ we have shown the vanishing of $H^1(M, E(\sigma_i))$.

4.8. We now look at the situation when $\underline{g}^{\mathbf{C}} \simeq \underline{g}_1 \oplus \cdots \oplus \underline{g}_r$ with $r \geq 2$, and where $\underline{g}_i$ are simple Lie algebra. $\underline{g}_i$ is one of the types considered above. We have the corresponding decompositions, $\underline{k}^{\mathbf{C}} \simeq \underline{k}_1 \oplus \cdots \oplus \underline{k}_r$ where $\underline{k}_i$ is the complexification of the Lie algebra of a maximal compact subalgebra of $\underline{g}_i$, and $\underline{p} \simeq \underline{p}_1^+ \oplus \cdots \oplus \underline{p}_r^+$, where $\underline{p}_i^+$ is the space of holomorphic tangent vectors corresponding to the symmetric space defined by $\underline{g}_i$, insider $\underline{g}^{\mathbf{C}}$.

Let σ be representation of $\underline{k}^{\mathbf{C}}$ on any one of its simple components, which we can assume without loss of generality is a simple component of $\underline{k}_1$. Since α is assumed to be an irreducible component of $\underline{p}^+, \alpha$ can be any $\underline{p}_i^+$, thought of as a $\underline{k}^{\mathbf{C}}$-module. We note that with respect to the metrics we are considering, there is a decomposition of the Casimir of $\underline{g}^{\mathbf{C}}$ as a sum of the Casimir corresponding to the $\underline{g}_i$. We now calculate the constants D.

i) α is the irreducible representation of $\underline{k}^{\mathbf{C}}$ on $\underline{p}_1^+$. In this case the values of the Casimirs of $\underline{k}_i$ $(i \geq 2)$ are zero, and we are thus reduced to the inequalities concerning the first cohomology of the holomorphic vector bundle on cocompact quotients of G_1/K_1 corresponding to the representation σ of K_1.

ii) α is $\underline{k}^{\mathbf{C}}$-representation $\underline{p}_i, i \neq 1$, then $\sigma \otimes \alpha$ is an irreducible representation of $\underline{k}^{\mathbf{C}}(\sigma \oplus \alpha)(C) = \sigma(C_1) + \alpha_i(C_i)$.

$$D = \frac{1}{4}\frac{\sigma(C_L^1)+\alpha_i(C_L^i)-\sigma(C_L^1)+\alpha_i(C_L^i)}{\sigma(C_k^i)+\alpha_i(C_K^1)-\sigma(C_K^1)}$$

$$D = \frac{1}{4}\frac{2\alpha_i(C_L^i)}{\alpha_i(C_k^i)}$$

We are thus reduced to the inequalities concerning the first Betti number of lattices in a simple group having Lie algebra $\underline{g}_i$. We see from (3.11) and the calculations of this chapter, that when $\underline{g}$ is simple, the first Betti number of M vanishes whenever the complex structure on M is also rigid. Thus in the situation when $\underline{g}$ is not simple, the positivity of $D + \lambda_1$ reduces to showing it for the simple components of $\underline{g}$.

4.9. Summarising, we have from (43), (44), (45), (46), (42), (49), (50), (51), that the following rigidity result holds:

Theorem 8. *Assume that G is such that the simple components of G are not of the type $I_{m,1} (m \geq 1)$, $I_{2,2} \simeq IV_4$, $II_3 \simeq I_{3,1}$, $III_2 \simeq IV_3$,. Then the complex structure on $\Gamma\backslash G/K'$ is infinitesimally rigid. In particular the complex structure on $\Gamma\backslash G/T$ is infinitesimally rigid.*

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